Insights from Number Theory: From Arithmetic Geometry to Quantum Topology

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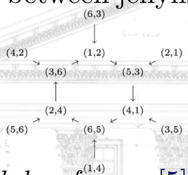
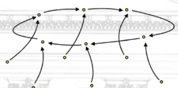
This defense concerns interconnected discoveries spanning the subjects of the representation theory of the symmetric group, the arithmetic-geometric mean and its finite-field analogue, and newly defined invariants of 3-manifolds, all tied together by the theory of modular forms.

We first derive asymptotic formulas for two families of zeros within the character tables of the symmetric groups, focusing on those indexed by ℓ-core partitions for primes $\ell \geq 5$. These results answer a question of McKay and shed light on the properties of large ℓ-core partitions. These results originally appeared in a joint paper with Ono [4].

We also revisit the classical arithmetic-geometric mean through a modern lens by investigating a finite field analogue first defined and studied by Griffin, Ono, Saikia, and Tsai. [2]. In drawing parallels with the work of Gauss on the classical AGM, we uncover the underlying structure of “jellyfish swarms” – directed graphs that organize the finite field AGM. These swarms serve as a novel framework for organizing Legendre elliptic curves over finite fields. This exploration yields new identities for Gauss’ class numbers and offers insights into the interplay between jellyfish sizes and the orders of certain elements in intimately related class groups.





Lastly, we look at *quantum modular forms* [5], which surprisingly arise in the study of 3-manifold invariants. Inspired by the pioneering work of Lawrence and Zagier, we give infinite families of quantum modular invariants. We use a recently developed theory of Akhmechet, Johnson, and Krushkal (AJK) which extends lattice cohomology and BPS q-series of 3-manifolds [1]. In particular, we prove that if Σ is a Brieskorn sphere and t is a root of unity, then their invariant $\widehat{\widehat{Z}}_{\Sigma}(t, q)$ is a quantum modular form. These results appeared in a joint paper with Liles [3].

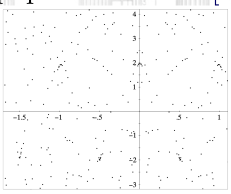


Figure 1. Graph of $\Re(f(x))$

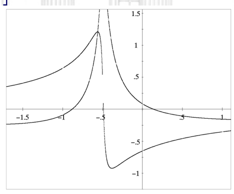


Figure 2. Graph of $\Re(h(x))$ and $\Im(h(x))$

References

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[2] M. J. Griffin, K. Ono, N. Saikia, and W-L. Tsai, “AGM and jellyfish swarms of elliptic curves,” *The American Mathematical Monthly.* DOI: 10.1080/00029890.2022.2160157, 2023, Taylor & Francis.

[3] L. Liles and E. McSpirit, “Infinite families of quantum modular 3-manifold invariants,” *arXiv:2306.14765*, 2023.

[4] E. McSpirit and K. Ono, “Zeros in the character tables of symmetric groups with an ℓ-core index,” *Canadian Mathematical Bulletin*, vol. 66, no. 2, pp. 467–476, 2023, Canadian Mathematical Society.

[5] D. Zagier, “Quantum modular forms,” *Quanta of maths*, vol. 11, pp. 659-675, 2010.

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